METAPLECTIC REPRESENTATION ON WIENER AMALGAM SPACES AND APPLICATIONS TO THE SCHRÖDINGER EQUATION

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ABSTRACT. We study the action of metaplectic operators on Wiener amalgam spaces, giving upper bounds for their norms. As an application, we obtain new fixed-time estimates in these spaces for Schrödinger equations with general quadratic Hamiltonians and Strichartz estimates for the Schrödinger equation with potentials $V(x) = \pm|x|^2$.

1. INTRODUCTION

The Wiener amalgam spaces were introduced by Feichtinger [10] in 1980 and soon they revealed to be, together with the closely related modulation spaces, the natural framework for the Time-Frequency Analysis; see e.g. [11, 12, 14, 15] and Gröchenig's book [18]. These spaces are modeled on the L^p spaces but they turn out to be much more flexible, since they control the local regularity of a function and its decay at infinity separately. For example, the Wiener amalgam space $W(B, L^q)$, $B = L^p$ or $B = \mathcal{F}L^p$, etc., consists of functions which locally have the regularity of a function in B but globally display a L^q decay.

In this paper we focus our attention on the action of the metaplectic representation on Wiener amalgam spaces. The metaplectic representation $\mu : Sp(d, \mathbb{R}) \to \mathcal{U}(L^2(\mathbb{R}^d))$ of the symplectic group $Sp(d, \mathbb{R})$ (see the subsequent Section 2 and [16] for details), was first constructed by Segal and Shale [26, 27] in the framework of quantum mechanics (though on the algebra level the first construction is due to van Hove [38]) and by Weil [39] in number theory. Since then, the metaplectic representation has attracted the attention of many people in different fields of mathematics and physics. In particular, we highlight the applications in the framework of reproducing formulae and wavelet theory [6], frame theory [13], quantum mechanics [9] and PDE's [21, 22].

Fix a test function $g \in \mathcal{C}_0^\infty$ and $1 \leq p, q \leq \infty$. Then, the *Wiener amalgam space* $W(\mathcal{F}L^p, L^q)$ with local component $\mathcal{F}L^p$ and global component L^q is defined as the

space of all functions/tempered distributions such that

$$\|f\|_{W(\mathcal{F}L^p, L^q)} := \|\|\|fT_x g\|_{\mathcal{F}L^p}\|_{L^q_x} < \infty,$$

where $T_x g(t) := g(t - x)$. To give a flavor of the type of results:

If $1 \leq p \leq q \leq \infty$ and $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d, \mathbb{R})$, with $\det B \neq 0$, then the meta-plectic operator $\mu(\mathcal{A})$ is a continuous mapping from $W(\mathcal{F}L^q, L^p)$ into $W(\mathcal{F}L^p, L^q)$, that is

$$\|\mu(\mathcal{A})f\|_{W(\mathcal{F}L^p, L^q)} \leq \alpha(\mathcal{A}, p, q)\|f\|_{W(\mathcal{F}L^q, L^p)}.$$

The norm upper bound $\alpha = \alpha(\mathcal{A}, p, q)$ is explicitly expressed in terms of the matrix $\mathcal{A}$ and the indices p, q (see Theorems 4.1 and 4.2).

This analysis generalize the basic result [11]:

The Fourier transform $\mathcal{F}$ is a continuous mapping between $W(\mathcal{F}L^q, L^p)$ and $W(\mathcal{F}L^p, L^q)$ if (and only if) $1 \leq p \leq q \leq \infty$.

Indeed, the Fourier transform $\mathcal{F}$ is a special metaplectic operator. If we introduce the symplectic matrix

$$(1) \qquad\qquad J = \begin{bmatrix} 0 & I_d \\ -I_d & 0 \end{bmatrix},$$

then $\mathcal{F}$ is (up to a phase factor) the unitary metaplectic operator corresponding to J,

$$\mu(J) = (-i)^{d/2} \mathcal{F}.$$

A fundamental tool to achieve these estimates is represented by the analysis of the dilation operator $f(x) \mapsto f(Ax)$, for a real invertible $d \times d$ matrix $A \in GL(d, \mathbb{R})$, with bounds on its norm in terms of spectral invariants of A. In the framework of modulation spaces such an investigation was recently developed in the scalar case $A = \lambda I$ by Sugimoto and Tomita [31, 32]. In Section 3 we study this problem for a general matrix $A \in GL(d, \mathbb{R})$ for both modulation and Wiener amalgam spaces. In particular, we extend the results in [31] to the case of a symmetric matrix A.

In the second part of the paper we present some natural applications to partial differential equations *with variable coefficients*. Precisely, we study the Cauchy problem for the Schrödinger equation with a quadratic Hamiltonian, namely

$$(2) \qquad\qquad \begin{cases} i\dfrac{\partial u}{\partial t} + Hu = 0 \\ u(0, x) = u_0(x), \end{cases}$$

where H is the Weyl quantization of a quadratic form on $\mathbb{R}^d \times \mathbb{R}^d$. The most interesting case is certainly the Schrödinger equation with a quadratic potential.

Indeed, the solution $u(t,x)$ to (2) is given by

$$u(t,x) = e^{itH}u_0,$$

where the operator e^{itH} is a metaplectic operator, so that the estimates resulting from the previous sections provide at once fixed-time estimates for the solution $u(t,x)$, in terms of the initial datum u_0. An example is provided by the Harmonic Oscillator $H = -\frac{1}{4\pi}\Delta + \pi|x|^2$ (see, e.g., [16, 20, 25]), for which we deduce the dispersive estimate

(3) $$\|e^{itH}u_0\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim |\sin t|^{-d}\|u_0\|_{W(\mathcal{F}L^\infty,L^1)}.$$

Another Hamiltonian we take into account is $H = -\frac{1}{4\pi}\Delta - \pi|x|^2$ (see [4]). In this case, we show

(4) $$\|e^{itH_A}u_0\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim \left(\frac{1 + |\sinh t|}{\sinh^2 t}\right)^{\frac{d}{2}}\|u_0\|_{W(\mathcal{F}L^\infty,L^1)}.$$

In Section 5 we shall combine these estimates with orthogonality arguments as in [7, 24] to obtain space-time estimates: the so-called Strichartz estimates (for the classical theory in Lebesgue spaces, see [17, 23, 24, 40]). For instance, the homogeneous Strichartz estimates achieved for the Harmonic Oscillator $H = -\frac{1}{4\pi}\Delta + \pi|x|^2$ read

$$\|e^{itH}u_0\|_{L^{q/2}([0,T])W(\mathcal{F}L^{r'},L^r)_x} \lesssim \|u_0\|_{L^2_x},$$

for every $T > 0$, $4 < q, \tilde{q} \leq \infty$, $2 \leq r, \tilde{r} \leq \infty$, such that $2/q + d/r = d/2$, and, similarly, for $\tilde{q}, \tilde{r}$. In the endpoint case $(q,r) = (4, 2d/(d-1))$, $d > 1$, we prove the same estimate with $\mathcal{F}L^{r'}$ replaced by the slightly larger $\mathcal{F}L^{r',2}$, where $L^{r',2}$ is a Lorentz space (Theorem 5.2).

The case of the Hamiltonian $H = -\frac{1}{4\pi}\Delta - \pi|x|^2$ will be detailed in Subsection 5.2. Finally, we shall compare all these estimates with the classical ones in the Lebesgue spaces (Subsection 5.3).

Our analysis combines techniques from time-frequency analysis (e.g., convolution relations, embeddings and duality properties of Wiener amalgam and modulation spaces) with methods from classical harmonic analysis and PDE's theory (interpolation results, Hölder-type inequalities, fractional integration theory).

This study carries on the one in [7], developed for the usual Schrödinger equation ($H = -\Delta$).

We record that hybrid spaces like the Wiener amalgam ones had appeared before as a technical tool in PDE's (see, e.g., Tao [33]). Notice that fixed-time estimates between modulation spaces in the case $H = -\Delta$ were also considered in [1, 2, 3].

Finally we observe that, by combining the Strichartz estimates in the present paper with arguments from functional analysis as in [8], wellposedness for Schrödinger

equations as above can also be proved, with an additional potential term in suitable Wiener amalgam spaces (see also [7, Section 6]). However, here we do not give details on this subject, that will be studied in a subsequent paper.

Notation. We define $|x|^2 = x \cdot x$, for $x \in \mathbb{R}^d$, where $x \cdot y = xy$ is the scalar product on $\mathbb{R}^d$. The space of smooth functions with compact support is denoted by $\mathcal{C}_0^\infty(\mathbb{R}^d)$, the Schwartz class is $\mathcal{S}(\mathbb{R}^d)$, the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$. The Fourier transform is normalized to be $\hat{f}(\xi) = \mathcal{F}f(\xi) = \int f(t)e^{-2\pi it\xi}dt$. Translation and modulation operators (*time and frequency shifts*) are defined, respectively, by

$$T_x f(t) = f(t - x) \quad \text{and} \quad M_\xi f(t) = e^{2\pi i\xi t}f(t).$$

We have the formulas $(T_x f)\widehat{\ } = M_{-x}\hat{f}$, $(M_\xi f)\widehat{\ } = T_\xi \hat{f}$, and $M_\xi T_x = e^{2\pi ix\xi}T_x M_\xi$. The notation $A \lesssim B$ means $A \leq cB$ for a suitable constant $c > 0$, whereas $A \asymp B$ means $c^{-1}A \leq B \leq cA$, for some $c \geq 1$. The symbol $B_1 \hookrightarrow B_2$ denotes the continuous embedding of the linear space B_1 into B_2.

2. Function spaces and preliminaries

2.1. **Lorentz spaces.** ([29, 30]). We recall that the Lorentz space $L^{p,q}$ on $\mathbb{R}^d$ is defined as the space of temperate distributions f such that

$$\|f\|_{pq}^* = \left(\frac{q}{p}\int_0^\infty [t^{1/p}f^*(t)]^q \frac{dt}{t}\right)^{1/q} < \infty,$$

when $1 \leq p < \infty$, $1 \leq q < \infty$, and

$$\|f\|_{pq}^* = \sup_{t>0} t^{1/p}f^*(t) < \infty$$

when $1 \leq p \leq \infty$, $q = \infty$. Here, as usual, $\lambda(s) = |\{|f| > s\}|$ denotes the distribution function of f and $f^*(t) = \inf\{s : \lambda(s) \leq t\}$.

One has $L^{p,q_1} \hookrightarrow L^{p,q_2}$ if $q_1 \leq q_2$, and $L^{p,p} = L^p$. Moreover, for $1 < p < \infty$ and $1 \leq q \leq \infty$, $L^{p,q}$ is a normed space and its norm $\|\cdot\|_{L^{p,q}}$ is equivalent to the above quasi-norm $\|\cdot\|_{pq}^*$.

We now recall the following generalized Hardy-Littlewood-Sobolev fractional integration theorem (see e.g. [28, page 119] and [37, Theorem 2, page 139]), which will be used in the sequel (the original fractional integration theorem corresponds to the model case of convolution by $K(x) = |x|^{-\alpha} \in L^{d/\alpha,\infty}$, $0 < \alpha < d$).

Proposition 2.1. *Let $1 \leq p < q < \infty$, $0 < \alpha < d$, with $1/p = 1/q + 1 - \alpha/d$. Then,*

$$(5) \qquad\qquad L^p(\mathbb{R}^d) * L^{d/\alpha,\infty}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d).$$

2.2. **Wiener amalgam spaces.** ([10, 11, 12, 14, 15]). Let $g \in \mathcal{C}_0^\infty$ be a test function that satisfies $\|g\|_{L^2} = 1$. We will refer to g as a window function. Let B one of the following Banach spaces: $L^p, \mathcal{F}L^p$, $1 \leq p \leq \infty$, $L^{p,q}$, $1 < p < \infty$, $1 \leq q \leq \infty$, possibly valued in a Banach space, or also spaces obtained from these by real or complex interpolation. Let C be one of the following Banach spaces: L^p, $1 \leq p \leq \infty$, or $L^{p,q}$, $1 < p < \infty$, $1 \leq q \leq \infty$, scalar-valued. For any given function f which is locally in B (i.e. $gf \in B$, $\forall g \in \mathcal{C}_0^\infty$), we set $f_B(x) = \|fT_x g\|_B$.

The *Wiener amalgam space* $W(B, C)$ with local component B and global component C is defined as the space of all functions f locally in B such that $f_B \in C$. Endowed with the norm $\|f\|_{W(B,C)} = \|f_B\|_C$, $W(B, C)$ is a Banach space. Moreover, different choices of $g \in \mathcal{C}_0^\infty$ generate the same space and yield equivalent norms.

If $B = \mathcal{F}L^1$ (the Fourier algebra), the space of admissible windows for the Wiener amalgam spaces $W(\mathcal{F}L^1, C)$ can be enlarged to the so-called Feichtinger algebra $W(\mathcal{F}L^1, L^1)$. Recall that the Schwartz class $\mathcal{S}$ is dense in $W(\mathcal{F}L^1, L^1)$.

We use the following definition of mixed Wiener amalgam norms. Given a measurable function F of the two variables (t, x) we set

$$\|F\|_{W(L^{q_1}, L^{q_2})_t W(\mathcal{F}L^{r_1}, L^{r_2})_x} = \left\| \|F(t, \cdot)\|_{W(\mathcal{F}L^{r_1}, L^{r_2})_x} \right\|_{W(L^{q_1}, L^{q_2})_t}.$$

Observe that [7]

$$\|F\|_{W(L^{q_1}, L^{q_2})_t W(\mathcal{F}L^{r_1}, L^{r_2})_x} = \|F\|_{W\left(L_t^{q_1}(W(\mathcal{F}L_x^{r_1}, L_x^{r_2})), L_t^{q_2}\right)}.$$

The following properties of Wiener amalgam spaces will be frequently used in the sequel.

Lemma 2.1. *Let B_i, C_i, $i = 1, 2, 3$, be Banach spaces such that $W(B_i, C_i)$ are well defined. Then,*

(i) Convolution. *If $B_1 * B_2 \hookrightarrow B_3$ and $C_1 * C_2 \hookrightarrow C_3$, we have*

(6) $$W(B_1, C_1) * W(B_2, C_2) \hookrightarrow W(B_3, C_3).$$

In particular, for every $1 \leq p, q \leq \infty$, we have

(7) $$\|f * u\|_{W(\mathcal{F}L^p, L^q)} \leq \|f\|_{W(\mathcal{F}L^\infty, L^1)} \|u\|_{W(\mathcal{F}L^p, L^q)}.$$

(ii) Inclusions. *If $B_1 \hookrightarrow B_2$ and $C_1 \hookrightarrow C_2$,*

$$W(B_1, C_1) \hookrightarrow W(B_2, C_2).$$

Moreover, the inclusion of B_1 into B_2 need only hold "locally" and the inclusion of C_1 into C_2 "globally". In particular, for $1 \leq p_i, q_i \leq \infty$, $i = 1, 2$, we have

(8) $$p_1 \geq p_2 \text{ and } q_1 \leq q_2 \implies W(L^{p_1}, L^{q_1}) \hookrightarrow W(L^{p_2}, L^{q_2}).$$

(iii) Complex interpolation. *For $0 < \theta < 1$, we have*

$$[W(B_1, C_1), W(B_2, C_2)]_{[\theta]} = W\left([B_1, B_2]_{[\theta]}, [C_1, C_2]_{[\theta]}\right),$$

if C_1 or C_2 has absolutely continuous norm.

(iv) Duality. *If B', C' are the topological dual spaces of the Banach spaces B, C respectively, and the space of test functions $\mathcal{C}_0^\infty$ is dense in both B and C, then*

(9) $$W(B, C)' = W(B', C').$$

Proposition 2.2. *For every $1 \le p \le q \le \infty$, the Fourier transform $\mathcal{F}$ maps $W(\mathcal{F}L^q, L^p)$ in $W(\mathcal{F}L^p, L^q)$ continuously.*

The proof of all these results can be found in ([10, 11, 12, 19]).

The subsequent result of real interpolation is proved in [7].

Proposition 2.3. *Given two local components B_0, B_1 as above, for every $1 \le p_0, p_1 < \infty$, $0 < \theta < 1$, $1/p = (1-\theta)/p_0 + \theta/p_1$, and $p \le q$ we have*

$$W\left((B_0, B_1)_{\theta,q}, L^p\right) \hookrightarrow \left(W(B_0, L^{p_0}), W(B_1, L^{p_1})\right)_{\theta,q}.$$

2.3. **Modulation spaces.** ([18]). Let $g \in \mathcal{S}$ be a non-zero window function. The short-time Fourier transform (STFT) $V_g f$ of a function/tempered distribution f with respect to the the window g is defined by

$$V_g f(z, \xi) = \int e^{-2\pi i \xi y} f(y) g(y - z)\, dy,$$

i.e., the Fourier transform $\mathcal{F}$ applied to $f T_z g$.

For $1 \le p, q \le \infty$, the modulation space $M^{p,q}(\mathbb{R}^n)$ is defined as the space of measurable functions f on $\mathbb{R}^n$ such that the norm

$$\|f\|_{M^{p,q}} = \|\|V_g f(\cdot, \xi)\|_{L^p}\|_{L^q_\omega}$$

is finite. Among the properties of modulation spaces, we record that $M^{2,2} = L^2$, $M^{p_1,q_1} \hookrightarrow M^{p_2,q_2}$, if $p_1 \le p_2$ and $q_1 \le q_2$. If $p, q < \infty$, then $(M^{p,q})' = M^{p',q'}$.

For comparison, notice that the norm in the Wiener amalgam spaces $W(\mathcal{F}L^p, L^q)$ reads

$$\|f\|_{W(\mathcal{F}L^p, L^q)} = \|\|V_g f(z, \cdot)\|_{L^p}\|_{L^q_z}.$$

The relationship between modulation and Wiener amalgam spaces is expressed by the following result.

Proposition 2.4. *The Fourier transform establishes an isomorphism $\mathcal{F} : M^{p,q} \to W(\mathcal{F}L^p, L^q)$.*

Consequently, convolution properties of modulation spaces can be translated into point-wise multiplication properties of Wiener amalgam spaces, as shown below.

Proposition 2.5. *For every* $1 \leq p, q \leq \infty$ *we have*

$$\|fu\|_{W(\mathcal{F}L^p, L^q)} \leq \|f\|_{W(\mathcal{F}L^1, L^\infty)} \|u\|_{W(\mathcal{F}L^p, L^q)}.$$

Proof. From Proposition 2.4, the estimate to prove is equivalent to

$$\|\hat{f} * \hat{u}\|_{M^{p,q}} \leq \|\hat{f}\|_{M^{1,\infty}} \|\hat{u}\|_{M^{p,q}},$$

but this a special case of [5, Proposition 2.4]. $\qquad\square$

The characterization of the $M^{2,\infty}$-norm in [31, Lemma 3.3] can be rephrased in our context as follows.

Lemma 2.2. *Suppose that* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *is a real-valued function satisfying* $\varphi = 1$ *on* $[-1/2, 1/2]^d$, $\operatorname{supp} \varphi \subset [-1, 1]^d$, $\varphi(t) = \phi(-t)$ *and* $\sum_{k \in \mathbb{Z}^d} \varphi(t - k) = 1$ *for all* $t \in \mathbb{R}^d$. *Then*

$$\tag{10} \|f\|_{M^{2,\infty}} \asymp \sup_{k \in \mathbb{Z}^d} \|(M_k \Phi) * f\|_{L^2},$$

for all $f \in M^{2,\infty}$, *where* $\Phi = \mathcal{F}^{-1}\varphi$.

To compute the $M^{p,q}$-norm we shall often use the *duality* technique, justified by the result below (see [18, Proposition 11.3.4 and Theorem 11.3.6] and [31, Relation (2.1)]).

Lemma 2.3. *Let* $\varphi \in \mathcal{S}(\mathbb{R}^d)$, *with* $\|\varphi\|_2 = 1$, $1 \leq p, q < \infty$. *Then* $(M^{p,q})^* = M^{p',q'}$, *under the duality*

$$\tag{11} \langle f, g \rangle = \langle V_\varphi f, V_\varphi g \rangle = \int_{\mathbb{R}^{2d}} V_\varphi f(x, \omega) \overline{V_\varphi g(x, \omega)} \, dx \, d\xi,$$

for $f \in M^{p,q}$, $g \in M^{p',q'}$.

Lemma 2.4. *Assume* $1 < p, q \leq \infty$ *and* $f \in M^{p,q}$. *Then*

$$\tag{12} \|f\|_{M^{p,q}} = \sup_{\|g\|_{M^{p',q'}} \leq 1} |\langle f, g \rangle|.$$

Notice that (12) still holds true whenever $p = 1$ or $q = 1$ and $f \in \mathcal{S}(\mathbb{R}^d)$, simply by extending [18, Theorem 3.2.1] to the duality $_{\mathcal{S}'}\langle \cdot, \cdot \rangle_{\mathcal{S}}$.

Finally we recall the behaviour of modulation spaces with respect to complex interpolation (see [11, Corollary 2.3].

Proposition 2.6. *Let* $1 \leq p_1, p_2, q_1, q_2 \leq \infty$, *with* $q_2 < \infty$. *If* T *is a linear operator such that, for* $i = 1, 2$,

$$\|Tf\|_{M^{p_i,q_i}} \leq A_i \|f\|_{M^{p_i,q_i}} \quad \forall f \in M^{p_i,q_i},$$

then

$$\|Tf\|_{M^{p,q}} \leq C A_1^{1-\theta} A_2^\theta \|f\|_{M^{p,q}} \quad \forall f \in M^{p,q},$$

where $1/p = (1 - \theta)/p_1 + \theta/p_2$, $1/q = (1 - \theta)/q_1 + \theta/q_2$, $0 < \theta < 1$ *and* C *is independent of* T.

2.4. **The metaplectic representation.** ([16]). The symplectic group is defined by

$$Sp(d, \mathbb{R}) = \left\{ g \in GL(2d, \mathbb{R}) : {}^t g J g = J \right\},$$

where the symplectic matrix J is defined in (1). The metaplectic or Shale-Weil representation μ is a unitary representation of the (double cover of the) symplectic group $Sp(d, \mathbb{R})$ on $L^2(\mathbb{R}^d)$. For elements of $Sp(d, \mathbb{R})$ in special form, the metaplectic representation can be computed explicitly. For $f \in L^2(\mathbb{R}^d)$, we have

$$\mu \left(\begin{bmatrix} A & 0 \\ 0 & {}^t A^{-1} \end{bmatrix} \right) f(x) = (\det A)^{-1/2} f(A^{-1} x)$$

(13)
$$\mu \left(\begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \right) f(x) = \pm e^{i\pi \langle Cx, x \rangle} f(x).$$

The symplectic algebra $\mathfrak{sp}(d, \mathbb{R})$ is the set of all $2d \times 2d$ real matrices $\mathcal{A}$ such that $e^{t\mathcal{A}} \in Sp(d, \mathbb{R})$ for all $t \in \mathbb{R}$.

The following formulae for the metaplectic representation can be found in [16, Theorems 4.51 and 4.53].

Proposition 2.7. *Let* $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d, \mathbb{R})$.

(i) If $\det B \neq 0$ *then*

(14) $$\mu(\mathcal{A}) f(x) = i^{d/2} (\det B)^{-1/2} \int e^{-\pi i x \cdot DB^{-1} x + 2\pi i y \cdot B^{-1} x - \pi i y \cdot B^{-1} A y} f(y) \, dy.$$

(ii) If $\det A \neq 0$,

(15) $$\mu(\mathcal{A}) f(x) = (\det A)^{-1/2} \int e^{-\pi i x \cdot CA^{-1} x + 2\pi i \xi \cdot A^{-1} x + \pi i \xi \cdot A^{-1} B \xi} \hat{f}(\xi) \, d\xi.$$

The following hybrid formula will be also used in the sequel.

Proposition 2.8. *If* $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d, \mathbb{R})$, $\det B \neq 0$ *and* $\det A \neq 0$, *then*

(16) $$\mu(\mathcal{A}) f(x) = (-i \det B)^{-1/2} e^{-\pi i x \cdot CA^{-1} x} \left(e^{-\pi i y \cdot B^{-1} A y} * f \right) (A^{-1} x).$$

Proof. By (15) we can write

$$\mu(\mathcal{A}) f(x) = (\det A)^{-1/2} e^{-\pi i x \cdot CA^{-1} x} \int e^{2\pi i \xi \cdot A^{-1} x} \mathcal{F} \left(\mathcal{F}^{-1} e^{\pi i \xi \cdot A^{-1} B \xi} \right) \hat{f}(\xi) \, d\xi$$

$$= (-i \det B)^{-1/2} e^{-\pi i x \cdot CA^{-1} x} \int e^{2\pi i \xi \cdot A^{-1} x} \mathcal{F} \left(e^{-\pi i y \cdot B^{-1} A y} * f \right) (\xi) \, d\xi,$$

where we used the formula (see [16, Theorem 2, page 257])

$$\mathcal{F}^{-1}\left(e^{i\pi\xi\cdot A^{-1}B\xi}\right)(y) = (-i\det A^{-1}B)^{-1/2}e^{-\pi i y\cdot B^{-1}Ay}.$$

Hence, from the Fourier inversion formula we obtain (16). $\square$

3. Dilation of Modulation and Wiener Amalgam Spaces

Given a function f on $\mathbb{R}^d$ and $A \in GL(d,\mathbb{R})$, we set $f_A(t) = f(At)$. We also consider the unitary operator $\mathcal{U}_A$ on $L^2(\mathbb{R}^d)$ defined by

$$(17) \qquad \mathcal{U}_A f(t) = |\det A|^{1/2} f(At) = |\det A|^{1/2} f_A(t).$$

In this section we study the boundedness of this operator on modulation and Wiener amalgam spaces. We need the following three lemmata.

Lemma 3.1. *Let $A \in GL(d,\mathbb{R})$, $\varphi(t) = e^{-\pi|t|^2}$, then*

$$V_\varphi \varphi_A(x,\xi) = (\det(A^*A+I))^{-1/2} e^{-\pi(I-(A^*A+I)^{-1})x\cdot x} M_{-((A^*A+I)^{-1})x} e^{-\pi(A^*A+I)^{-1}\xi\cdot\xi}.$$

Proof. By definition of the STFT,

$$
\begin{aligned}
V_\varphi \varphi_A(x,\xi) &= \int_{\mathbb{R}^d} e^{-\pi Ay\cdot y} e^{-2\pi i\xi\cdot y} e^{-\pi(y-x)^2}\, dy \\
&= e^{-\pi|x|^2} \int_{\mathbb{R}^d} e^{-\pi(A^*A+I)y\cdot y + 2\pi x\cdot y} e^{-2\pi i\xi\cdot y}\, dy.
\end{aligned}
$$

Now, we rewrite the generalized Gaussian above using the translation and dilation operators, that is

$$e^{-\pi(A^*A+I)y\cdot y + 2\pi x\cdot y} = (\det(A^*A+I))^{-1/4}(T_{(A^*A+I)^{-1}x}\mathcal{U}_{(A^*A+I)^{1/2}})\varphi(y)$$

and use the properties $\mathcal{F}\mathcal{U}_B = \mathcal{U}_{(B^*)^{-1}}\mathcal{F}$, for every $B \in GL(d,\mathbb{R})$ and $\mathcal{F}T_x = M_{-x}\mathcal{F}$. Thereby,

$$
\begin{aligned}
V_\varphi \varphi_A(x,\xi) &= e^{-\pi(I-(A^*A+I)^{-1})x\cdot x}(\det(A^*A+I))^{-1/4}\mathcal{F}(T_{(A^*A+I)^{-1}x}\mathcal{U}_{(A^*A+I)^{1/2}}\varphi(\xi) \\
&= e^{-\pi(I-(A^*A+I)^{-1})x\cdot x}(\det(A^*A+I))^{-1/2} M_{-(A^*A+I)^{-1}x} e^{-\pi(A^*A+I)^{-1}\xi\cdot\xi},
\end{aligned}
$$

as desired. $\square$

The result below generalizes [36, Lemma 1.8], recaptured in the special case $A = \lambda I$, $\lambda > 0$.

Lemma 3.2. *Let $1 \le p,q \le \infty$, $A \in GL(d,\mathbb{R})$ and $\varphi(t) = e^{-\pi|t|^2}$. Then,*

$$(18) \qquad \|\varphi_A\|_{M^{p,q}} = p^{-d/(2p)} q^{-d/(2q)} |\det A|^{-1/p} (\det(A^*A+I))^{-(1-1/q-1/p)/2}.$$

Proof. Since the modulation space norm is independent of the choice of the window function, we choose the Gaussian φ, so that $\|\varphi_A\|_{M^{p,q}} \asymp \|V_\varphi \varphi_A\|_{L^{p,q}}$. Since

$$
\begin{aligned}
\int_{\mathbb{R}^d} e^{-\pi p (I-(A^*A+I)^{-1})x \cdot x}\, dx &= (\det(I-(A^*A+I)^{-1})^{-1/2} p^{-d/2} \\
&= p^{-d/2} |\det A|^{-1} (\det(A^*A+I))^{1/2}
\end{aligned}
$$

and, analogously, $\int_{\mathbb{R}^d} e^{-\pi q (A^*A+I)^{-1} \xi \cdot \xi}\, d\xi = (\det(A^*A+I))^{1/2} q^{-d/2}$, the result immediately follows from Lemma 3.1. $\qquad\square$

We record [18, Lemma 11.3.3]:

Lemma 3.3. *Let $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi, \psi, \gamma \in \mathcal{S}(\mathbb{R}^d)$. Then,*

$$
|V_\varphi f(x,\xi)| \leq \frac{1}{\langle \gamma, \psi \rangle} (|V_\psi f| * |V_\varphi \gamma|)(x,\xi) \quad \forall (x,\xi) \in \mathbb{R}^{2d}.
$$

The results above are the ingredients for the first dilation property of modulation spaces we are going to present.

Proposition 3.1. *Let $1 \leq p,q \leq \infty$ and $A \in GL(d,\mathbb{R})$. Then, for every $f \in M^{p,q}(\mathbb{R}^d)$,*

$$
\tag{19} \|f_A\|_{M^{p,q}} \lesssim |\det A|^{-(1/p-1/q+1)} (\det(I+A^*A))^{1/2} \|f\|_{M^{p,q}}.
$$

Proof. The proof follows the guidelines of [31, Lemma 3.1]. First, by a change of variable, the dilation is transferred from the function f to the window φ:

$$
V_\varphi f_A(x,\xi) = |\det A|^{-1} V_{\varphi_{A^{-1}}} f(Ax,(A^*)^{-1}\xi).
$$

Whence, performing the change of variables $Ax = u$, $(A^*)^{-1}\xi = v$,

$$
\|f_A\|_{M^{p,q}} = |\det A|^{-1} \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |V_{\varphi_{A^{-1}}} f(Ax,(A^*)^{-1}\xi)|^p\, dx \right)^{q/p} d\xi \right)^{1/q}
$$

$$
= |\det A|^{-(1/p-1/q+1)} \|V_{\varphi_{A^{-1}}}\|_{L^{p,q}}.
$$

Now, Lemma 3.3, written for $\psi(t) = \gamma(t) = \varphi(t) = e^{-\pi t^2}$, yields the following majorization

$$
|V_{\varphi_{A^{-1}}} f(x,\xi)| \leq \|\varphi\|_{L^2}^{-2} (|V_\varphi f| * |V_{\varphi_{A^{-1}}} \varphi|)(x,\xi).
$$

Finally, Young's Inequality and Lemma 3.2 provide the desired result:

$$
\begin{aligned}
\|f_A\|_{M^{p,q}} &\lesssim |\det A|^{-(1/p-1/q+1)} \| |V_\varphi f| * |V_{\varphi_{A^{-1}}} \varphi| \|_{L^{p,q}} \\
&\lesssim |\det A|^{-(1/p-1/q+1)} \|V_\varphi f\|_{L^{p,q}} \|V_{\varphi_{A^{-1}}} \varphi\|_{L^1} \\
&\asymp |\det A|^{-(1/p-1/q+1)} (\det(I+A^*A))^{1/2} \|f\|_{M^{p,q}}.
\end{aligned}
$$

$\qquad\square$

Proposition 3.1 generalizes [31, Lemma 3.1], that can be recaptured by choosing the matrix $A = \lambda I$, $\lambda > 0$.

Corollary 3.2. *Let* $1 \leq p, q \leq \infty$ *and* $A \in GL(d, \mathbb{R})$. *Then, for every* $f \in W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$,

$$(20) \qquad \|f_A\|_{W(\mathcal{F}L^p, L^q)} \lesssim |\det A|^{(1/p - 1/q - 1)}(\det(I + A^*A))^{1/2}\|f\|_{W(\mathcal{F}L^p, L^q)}.$$

Proof. It follows immediately from the relation between Wiener amalgam spaces and modulation spaces given by $W(\mathcal{F}L^p, L^q) = \mathcal{F}M^{p,q}$ and by the relation $\widehat{(f_A)} = |\det A|^{-1}(\hat{f})_{(A^*)^{-1}}$. $\square$

In what follows we give a more precise result about the behaviour of the operator norm $\|D_A\|_{M^{p,q} \to M^{p,q}}$ in terms of A, when A is a symmetric matrix, extending the diagonal case $A = \lambda I$, $\lambda > 0$ treated in [31]. We shall use the set and index terminology of the paper above. Namely, for $1 \leq p \leq \infty$, let p' be the conjugate exponent of p $(1/p + 1/p' = 1)$. For $(1/p, 1/q) \in [0, 1] \times [0, 1]$, we define the subsets

$$I_1 = \max(1/p, 1/p') \leq 1/q, \qquad I_1^* = \min(1/p, 1/p') \geq 1/q,$$

$$I_2 = \max(1/q, 1/2) \leq 1/p', \qquad I_2^* = \min(1/q, 1/2) \geq 1/p',$$

$$I_3 = \max(1/q, 1/2) \leq 1/p, \qquad I_3^* = \min(1/q, 1/2) \geq 1/p,$$

as shown in Figure 1:

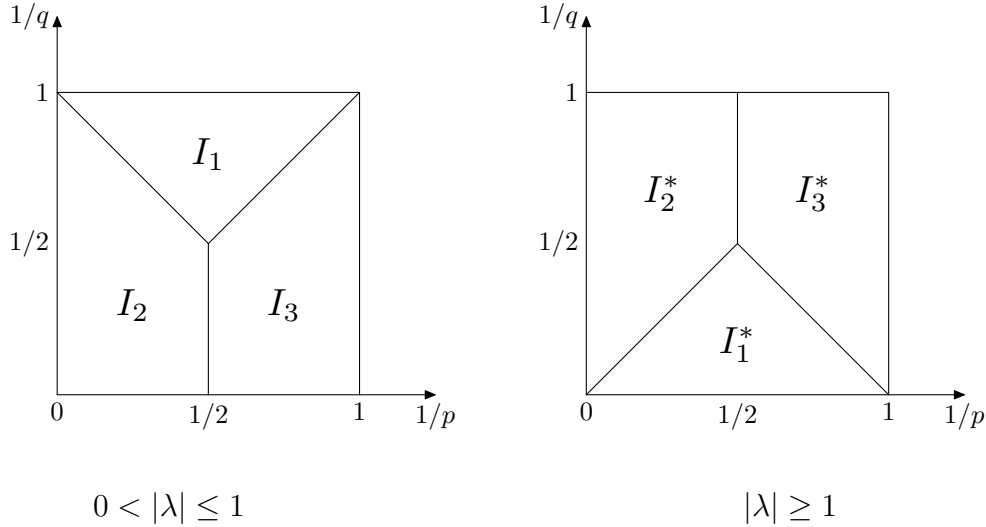

$$0 < |\lambda| \leq 1 \qquad\qquad |\lambda| \geq 1$$

Figure 1. The index sets.

We introduce the indices:

$$\mu_1(p,q) = \begin{cases} -1/p & \text{if } (1/p, 1/q) \in I_1^*, \\ 1/q - 1 & \text{if } (1/p, 1/q) \in I_2^*, \\ -2/p + 1/q & \text{if } (1/p, 1/q) \in I_3^*, \end{cases}$$

and

$$\mu_2(p,q) = \begin{cases} -1/p & \text{if } (1/p, 1/q) \in I_1, \\ 1/q - 1 & \text{if } (1/p, 1/q) \in I_2, \\ -2/p + 1/q & \text{if } (1/p, 1/q) \in I_3. \end{cases}$$

The above mentioned result by [31] reads as follows:

Theorem 3.3. *Let* $1 \le p, q \le \infty$, *and* $A = \lambda I$, $\lambda \ne 0$.
(i) We have

$$\|f_A\|_{M^{p,q}} \lesssim |\lambda|^{d\mu_1(p,q)} \|f\|_{M^{p,q}}, \qquad \forall \, |\lambda| \ge 1, \ \forall f \in M^{p,q}(\mathbb{R}^d).$$

Conversely, if there exists $\alpha > 0$ *such that*

$$\|f_A\|_{M^{p,q}} \lesssim |\lambda|^{\alpha} \|f\|_{M^{p,q}}, \qquad \forall \, |\lambda| \ge 1, \ \forall f \in M^{p,q}(\mathbb{R}^d),$$

then $\alpha \ge d\mu_1(p,q)$.
(ii) We have

$$\|f_A\|_{M^{p,q}} \lesssim |\lambda|^{d\mu_2(p,q)} \|f\|_{M^{p,q}}, \qquad \forall \, 0 < |\lambda| \le 1, \ \forall f \in M^{p,q}(\mathbb{R}^d).$$

Conversely, if there exists $\beta > 0$ *such that*

$$\|f_A\|_{M^{p,q}} \lesssim |\lambda|^{\beta} \|f\|_{M^{p,q}}, \qquad \forall \, 0 < |\lambda| \le 1, \ \forall f \in M^{p,q}(\mathbb{R}^d),$$

then $\beta \le d\mu_2(p,q)$.

Here is our extension.

Theorem 3.4. *Let* $1 \le p, q \le \infty$. *There exists a constant* $C > 0$ *such that, for every symmetric matrix* $A \in GL(d, \mathbb{R})$, *with eigenvalues* $\lambda_1, \dots, \lambda_d$, *we have*

$$(21) \qquad \|f_A\|_{M^{p,q}} \le C \prod_{j=1}^{d} (\max\{1, |\lambda_j|\})^{\mu_1(p,q)} (\min\{1, |\lambda_j|\})^{\mu_2(p,q)} \|f\|_{M^{p,q}},$$

for every $f \in M^{p,q}(\mathbb{R}^d)$.
Conversely, if there exist $\alpha_j > 0, \beta_j > 0$ *such that, for every* $\lambda_j \ne 0$,

$$\|f_A\|_{M^{p,q}} \le C \prod_{j=1}^{d} (\max\{1, |\lambda_j|\})^{\alpha_j} (\min\{1, |\lambda_j|\})^{\beta_j} \|f\|_{M^{p,q}}, \qquad \forall f \in M^{p,q}(\mathbb{R}^d),$$

with $A = \text{diag}[\lambda_1, \dots, \lambda_d]$, *then* $\alpha_j \ge \mu_1(p,q)$ *and* $\beta_j \le \mu_2(p,q)$.

Proof. The necessary conditions are an immediate consequence of the one dimensional case, already contained in Theorem 3.3. Indeed, it can be seen by taking f as tensor product of functions of one variable and by leaving free to vary just one eigenvalue, the remaining eigenvalues being all equal to one.

Let us come to the first part of the theorem. It suffices to prove it in the diagonal case $A = D = \text{diag}[\lambda_1, \ldots, \lambda_d]$. Indeed, since A is symmetric, there exists an orthogonal matrix T such that $A = T^{-1}DT$, and D is a diagonal matrix. On the other hand, by Proposition 3.1, we have $\|f_A\|_{M^{p,q}} \lesssim \|f_{T^{-1}D}\|_{M^{p,q}} = \|(f_{T^{-1}})_D\|_{M^{p,q}}$ and $\|f_{T^{-1}}\|_{M^{p,q}} \lesssim \|f\|_{M^{p,q}}$; hence the general case in (21) follows from the diagonal case $A = D$, with f replaced by $f_{T^{-1}}$.

From now onward, $A = D = \text{diag}[\lambda_1, \ldots, \lambda_d]$.

If the theorem holds true for a pair (p, q), with $(1/p, 1/q) \in [0, 1] \times [0, 1]$, then it is also true for their dual pair (p', q') (with $f \in \mathcal{S}$ if $p' = 1$ or $q' = 1$, see (12)). Indeed,

$$
\begin{aligned}
\|f_D\|_{M^{p',q'}} &= \sup_{\|g\|_{M^{p,q}} \le 1} |\langle f_D, g \rangle| = |\det D|^{-1} \sup_{\|g\|_{M^{p,q}} \le 1} |\langle f, g_{D^{-1}} \rangle| \\
&\le |\det D|^{-1} \|f\|_{M^{p',q'}} \sup_{\|g\|_{M^{p,q}} \le 1} \|g_{D^{-1}}\|_{M^{p,q}} \\
&\lesssim \prod_{j=1}^{d} |\lambda_j|^{-1} \prod_{j=1}^{d} (\max\{1, |\lambda_j|^{-1}\})^{\mu_1(p,q)} (\min\{1, |\lambda_j|^{-1}\})^{\mu_2(p,q)} \|f\|_{M^{p',q'}} \\
&= \prod_{j=1}^{d} (\max\{1, |\lambda_j|\})^{\mu_1(p',q')} (\min\{1, |\lambda_j|\})^{\mu_2(p',q')} \|f\|_{M^{p',q'}},
\end{aligned}
$$

for the index functions μ_1 and μ_2 fulfill:

$$
(22) \qquad \mu_1(p', q') = -1 - \mu_2(p, q), \quad \mu_2(p', q') = -1 - \mu_1(p, q).
$$

Hence it suffices to prove the estimate (21) for the case $p \ge q$. Notice that the estimate in $M^{1,q'}$, $q' > 1$, are proved for Schwartz functions only, but they extend to all functions in $M^{1,q'}$, $q' < \infty$, for $\mathcal{S}(\mathbb{R}^d)$ is dense in $M^{1,q'}$. The uncovered case $(1, \infty)$ will be verified directly at the end of the proof.

From Figure 1 it is clear that the estimate (21) for the points in the upper triangles follows by complex interpolation (Proposition 2.6) from the diagonal case $p = q$, and the two cases $(p, q) = (\infty, 1)$ and $(p, q) = (2, 1)$.
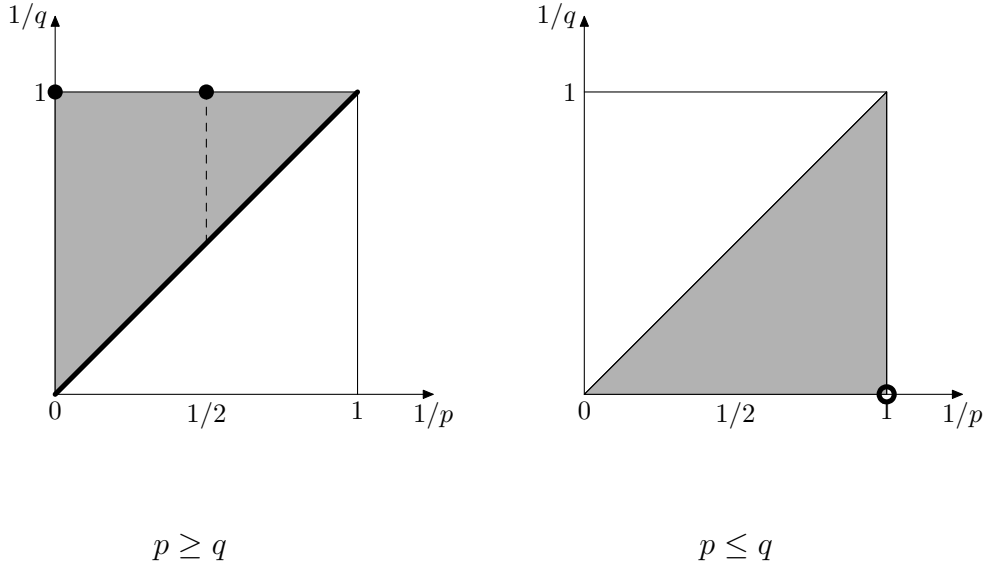
Figure 2. The complex interpolation and the duality method.

Case $p = q$. If $d = 1$ the claim is true by Theorem 3.3 in dimension $d = 1$. We then use the induction method. Namely, we assume that (21) is fulfilled in dimension $d - 1$ and prove that still holds in dimension d.

For $x, \xi \in \mathbb{R}^d$, we write $x = (x', x_d)$, $\xi = (\xi', \xi_d)$, with $x', \xi' \in \mathbb{R}^{d-1}$, $x_d, \xi_d \in \mathbb{R}$, $D' = \mathrm{diag}[\lambda_1, \ldots, \lambda_{d-1}]$, and choose the Gaussian $\varphi(x) = e^{-\pi|x|^2} = e^{-\pi|x'|^2} e^{-\pi|x_d|^2} = \varphi'(x') \varphi_d(x_d)$ as window function. Observe that $V_\varphi f_D$ admits the two representations

$$
\begin{aligned}
V_\varphi f_D(x', x_d, \xi', \xi_d) &= \int_{\mathbb{R}^d} f(\lambda_1 t_1, \ldots, \lambda_d t_d) \overline{M_{\xi'} T_{x'} \varphi'(t') M_{\xi_d} T_{x_d} \varphi_d(t_d)} \, dt' dt_d \\
&= V_{\varphi'}((F_{x_d, \xi_d, \lambda_d})_{D'}) \\
&= V_{\varphi_d}((G_{x', \xi', D'})_{\lambda_d})
\end{aligned}
$$

where

$$
F_{x_d, \xi_d, \lambda_d}(t') = V_{\varphi_d}(f(t', \lambda_d \cdot))(x_d, \xi_d), \quad G_{x', \xi', D'}(t_d) = V_{\varphi'}(f(D' \cdot, t_d))(x', \xi').
$$

By the inductive hypothesis we have

$$\|f_D\|_{M^{p,p}(\mathbb{R}^d)} = \|V_\varphi f_D\|_{L^p(\mathbb{R}^{2d})}$$

$$= \left(\int_{\mathbb{R}^2} \left(\int_{\mathbb{R}^{2(d-1)}} |V_{\varphi'}((F_{x_d,\xi_d,\lambda_d})_{D'})(x',\xi')|^p dx' d\xi' \right) dx_d d\xi_d \right)^{1/p}$$

$$\lesssim \prod_{j=1}^{d-1} (\max\{1,|\lambda_j|\})^{\mu_1(p,p)} (\min\{1,|\lambda_j|\})^{\mu_2(p,p)}$$

$$\cdot \left(\int_{\mathbb{R}^{2d}} |V_{\varphi'}(F_{x_d,\xi_d,\lambda_d})(x',\xi')|^p \, dx \, d\xi \right)^{1/p}$$

$$= \prod_{j=1}^{d-1} (\max\{1,|\lambda_j|\})^{\mu_1(p,p)} (\min\{1,|\lambda_j|\})^{\mu_2(p,p)}$$

$$\cdot \left(\int_{\mathbb{R}^{2(d-1)}} \left(\int_{\mathbb{R}^2} |V_{\varphi_d}((G_{x',\xi',I})_{\lambda_d})(x_d,\xi_d)|^p \, dx_d \, d\xi_d \right) dx' d\xi' \right)^{1/p}$$

$$\lesssim \prod_{j=1}^{d} (\max\{1,|\lambda_j|\})^{\mu_1(p,p)} (\min\{1,|\lambda_j|\})^{\mu_2(p,p)} \|f\|_{M^{p,p}(\mathbb{R}^d)},$$

where in the last raw we used Theorem 3.3 for $d=1$.

Case $(p,q) = (2,1)$. First, we prove the case $(p,q) = (2,\infty)$ and then obtain the claim by duality as above, since $\mathcal{S}$ is dense in $M^{2,1}$. Namely, we want to show that

$$\|f_D\|_{M^{2,\infty}} \lesssim \prod_{j=1}^{d} \max\{1,|\lambda_j|\})^{-1/2} (\min\{1,|\lambda_j|\})^{-1} \|f\|_{M^{2,\infty}}, \quad \forall f \in M^{2,\infty}.$$

The arguments are similar to [31, Lemma 3.4]. We use the characterization of the $M^{2,\infty}$-norm in (10)

$$\|f_D\|_{M^{2,\infty}} \lesssim |\det D|^{-1/2} \sup_{k \in \mathbb{Z}^d} \|\varphi(D \cdot -k)\hat{f}\|_{L^2}$$

(23)
$$= |\det D|^{-1/2} \sup_{k \in \mathbb{Z}^d} \|\varphi(D \cdot -k) \left(\sum_{l \in \mathbb{Z}^d} \varphi(\cdot - l) \right) \hat{f}\|_{L^2}.$$

Observe that

$$\left| \varphi(Dt - k) \left(\sum_{l \in \mathbb{Z}^d} \varphi(t - l) \right) \hat{f}(t) \right|^2 \leq 4^d \sum_{l \in \mathbb{Z}^d} \left| \varphi(Dt - k)\varphi(t - l)\hat{f}(t) \right|^2$$

$$= 4^d \sum_{l \in \Lambda_k} \left| \varphi(Dt - k)\varphi(t - l)\hat{f}(t) \right|^2$$

where

$$\Lambda_k = \left\{ l \in \mathbb{Z}^d : |l_j - \frac{k_j}{\lambda_j}| \le 1 + \frac{1}{|\lambda_j|} \right\}$$

and

$$\#\Lambda_k \le C \prod_{j=1}^d \min\{1, |\lambda_j|\}^{-1}, \quad \forall k \in \mathbb{Z}^d$$

(C being a constant depending on d only). Since $|\lambda_j| = \max\{1, |\lambda_j|\} \min\{1, |\lambda_j|\}$, the expression on the right-hand side of (23) is dominated by

$$C' \prod_{j=1}^d (\max\{1, |\lambda_j|\})^{-1/2} (\min\{1, |\lambda_j|\})^{-1} \sup_{m \in \mathbb{Z}^d} \|(M_m \Phi) * f\|_{L^2}.$$

Thereby the norm equivalence (10) gives the desired estimate.

Case $(p, q) = (\infty, 1)$. We have to prove that

$$\|f_D\|_{M^{\infty,1}} \lesssim \prod_{j=1}^d \max\{1, |\lambda_j|\} \|f\|_{M^{\infty,1}}, \quad \forall f \in M^{\infty,1}.$$

This estimate immediately follows from (19), written for $A = D = \text{diag}[\lambda_1, \ldots, \lambda_d]$:

$$\|f_D\|_{M^{\infty,1}} \lesssim \prod_{j=1}^d (1 + \lambda_j^2)^{1/2} \lesssim \prod_{j=1}^d \max\{1, |\lambda_j|\} \|f\|_{M^{\infty,1}}.$$

Case $(p, q) = (1, \infty)$. We are left to prove that

$$\|f_D\|_{M^{1,\infty}} \lesssim \prod_{j=1}^d (\max\{1, |\lambda_j|\})^{-1} (\min\{1, |\lambda_j|\})^{-2} \|f\|_{M^{1,\infty}}, \quad \forall f \in M^{1,\infty}.$$

This is again the estimate (19), written for $A = D = \text{diag}[\lambda_1, \ldots, \lambda_d]$:

$$\|f_D\|_{M^{1,\infty}} \lesssim \prod_{j=1}^d |\lambda_j|^{-2} \prod_{j=1}^d \max\{1, |\lambda_j|\} \|f\|_{M^{1,\infty}}.$$

□

Corollary 3.5. *Let $1 \le p, q \le \infty$. There exists a constant $C > 0$ such that, for every symmetric matrix $A \in GL(d, \mathbb{R})$, with eigenvalues $\lambda_1, \ldots, \lambda_d$, we have*

$$(24) \quad \|f_A\|_{W(\mathcal{F}L^p, L^q)} \le C \prod_{j=1}^d (\max\{1, |\lambda_j|\})^{\mu_1(p', q')} (\min\{1, |\lambda_j|\})^{\mu_2(p', q')} \|f\|_{W(\mathcal{F}L^p, L^q)},$$

for every $f \in W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$.

Conversely, if there exist $\alpha_j > 0, \beta_j > 0$ such that, for every $\lambda_j \neq 0$,

$$\|f_A\|_{W(\mathcal{F}L^p, L^q)} \leq C \prod_{j=1}^{d} (\max\{1, |\lambda_j|\})^{\alpha_j} (\min\{1, |\lambda_j|\})^{\beta_j} \|f\|_{W(\mathcal{F}L^p, L^q)},$$

for every $f \in W(\mathcal{F}L^p, L^q)(\mathbb{R}^d)$, with $A = \text{diag}[\lambda_1, \ldots, \lambda_d]$, then $\alpha_j \geq \mu_1(p', q')$ and $\beta_j \leq \mu_2(p', q')$.

Proof. It is a mere consequence of Theorem 3.4 and the index relation (22). Namely,

$$
\begin{aligned}
\|f_A\|_{W(\mathcal{F}L^p, L^q)} &= \|\widehat{f_A}\|_{M^{p,q}} = |\det A|^{-1} \|\hat{f}_{A^{-1}}\|_{M^{p,q}} \\
&\leq C \prod_{j=1}^{d} |\lambda_j|^{-1} \prod_{j=1}^{d} (\max\{1, |\lambda_j|^{-1}\})^{\mu_1(p,q)} (\min\{1, |\lambda_j|^{-1}\})^{\mu_2(p,q)} \|\hat{f}\|_{M^{p,q}} \\
&= C \prod_{j=1}^{d} (\max\{1, |\lambda_j|\})^{\mu_1(p',q')} (\min\{1, |\lambda_j|\})^{\mu_2(p',q')} \|f\|_{W(\mathcal{F}L^p, L^q)}.
\end{aligned}
$$

The necessary conditions use the same argument. $\qquad\square$

4. Action of metaplectic operators on Wiener amalgam spaces

In this section we study the continuity property of metaplectic operators on Wiener amalgam spaces, giving bounds on their norms. Here is our first result.

Theorem 4.1. *Let $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d, \mathbb{R})$, and $1 \leq p \leq q \leq \infty$.*

(i) If $\det B \neq 0$, then

$$(25) \qquad \|\mu(\mathcal{A})f\|_{W(\mathcal{F}L^p, L^q)} \lesssim \alpha(\mathcal{A}, p, q) \|f\|_{W(\mathcal{F}L^q, L^p)},$$

where

$$(26) \qquad \alpha(\mathcal{A}, p, q) = |\det B|^{1/q - 1/p - 3/2} |\det(I + B^*B)(B + iA)(B + iD)|^{1/2}.$$

(ii) If $\det A, \det B \neq 0$, then

$$(27) \qquad \|\mu(\mathcal{A})f\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \beta(\mathcal{A}) \|f\|_{W(\mathcal{F}L^\infty, L^1)},$$

with

$$(28) \qquad \beta(\mathcal{A}) = |\det A|^{-3/2} |\det B|^{-1} |\det(I + A^*A)(B + iA)(A + iC)|^{1/2}.$$

If the matrices A or B are symmetric, Theorem 4.1 can be sharpened as follows.

Theorem 4.2. *Let $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(d, \mathbb{R})$, and $1 \leq p \leq q \leq \infty$.*

(i) If $\det B \neq 0$, $B^* = B$, with eigenvalues $\lambda_1, \ldots, \lambda_d$, then

$$(29) \qquad \|\mu(\mathcal{A})f\|_{W(\mathcal{F}L^p, L^q)} \lesssim \alpha'(\mathcal{A}, p, q) \|f\|_{W(\mathcal{F}L^q, L^p)},$$

where

$$\alpha'(\mathcal{A}, p, q) = |\det(B + iA)(B + iD)|^{1/2}$$

$$(30) \qquad \cdot \prod_{j=1}^{d} (\max\{1, |\lambda_j|\})^{\mu_1(p,q)-1/2} (\min\{1, |\lambda_j|\})^{\mu_2(p,q)-1/2}.$$

(ii) If $\det A, \det B \neq 0$*, and* $A^* = A$ *with eigenvalues* $\nu_1, \ldots, \nu_d$*, then*

$$(31) \qquad \|\mu(\mathcal{A})f\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \beta'(\mathcal{A})\|f\|_{W(\mathcal{F}L^\infty, L^1)},$$

with

$$\beta'(\mathcal{A}) = |\det B|^{-1} |\det(B + iA)(A + iC)|^{1/2}$$

$$(32) \qquad \cdot \prod_{j=1}^{d} (\max\{1, |\nu_j|\})^{-1/2} (\min\{1, |\nu_j|\})^{-3/2}.$$

We now prove Theorems 4.1 and 4.2. We need the following preliminary result.

Lemma 4.1. *Let* R *be a* $d \times d$ *real symmetric matrix, and* $f(y) = e^{-\pi i R y \cdot y}$*. Then,*

$$(33) \qquad \|f\|_{W(\mathcal{F}L^1, L^\infty)} = |\det(I + iR)|^{1/2}.$$

Proof. We first compute the short-time Fourier transform of f, with respect to the window $g(y) = e^{-\pi|y|^2}$. We have

$$V_g f(x, \xi) = \int e^{-2\pi i y \xi} e^{-i\pi R y \cdot y} e^{-\pi|y|^2} \, dy$$

$$= e^{-\pi|x|^2} \int e^{-2\pi i y \cdot (\xi + ix) - \pi(I + iR)y \cdot y} \, dy$$

$$= e^{-\pi|x|^2} (\det(I + iR))^{-1/2} e^{-\pi(I+iR)^{-1}(\xi+ix)\cdot(\xi+ix)},$$

where we used [16, Theorem 1, page 256]. Hence

$$|V_g f(x, \xi)| = |\det(I + iR)|^{-1/2} e^{-\pi(I+R^2)^{-1}(\xi+Rx)\cdot(\xi+Rx)},$$

and, performing the change of variables $(I + R^2)^{-1/2}(\xi + Rx) = y$, with $d\xi = |\det(I + R^2)|^{1/2}dy$, we obtain

$$(34)$$
$$\int_{\mathbb{R}^d} V_g f(x, \xi) d\xi = |\det(I+iR)|^{-1/2} (\det(I+R^2))^{1/2} \int_{\mathbb{R}^d} e^{-\pi|y|^2} \, dy = |\det(I+iR)|^{1/2}.$$

The last equality follows from $(I+iR) = (I+R^2)(I-iR)^{-1}$, so that $\det(I+iR)^{-1} = \det(I + R^2)^{-1} \det(I - iR)$. Now, relation (33) is proved by taking the supremum with the respect to $x \in \mathbb{R}^d$ in (34).   □

Proof of Theorem 4.1. (i) We use the expression of $\mu(\mathcal{A})f$ in formula (14). The estimates below are obtained by using (in order): Proposition 2.5 with Lemma 4.1, the estimate (20), Proposition 2.2, and, finally, Proposition 2.5 combined with Lemma 4.1 again:

$$
\begin{aligned}
\|\mu(\mathcal{A})f\|_{W(\mathcal{F}L^p,L^q)} &= |\det B|^{-1/2} \left\| e^{-\pi i x \cdot DB^{-1}x} \mathcal{F}^{-1}\left(e^{-\pi i y \cdot B^{-1}Ay} f \right)(B^{-1}x) \right\|_{W(\mathcal{F}L^p,L^q)} \\
&\leq |\det B|^{-1/2} \| e^{-\pi i x \cdot DB^{-1}x} \|_{W(\mathcal{F}L^1,L^\infty)} \\
&\qquad \cdot \left\| \left(\mathcal{F}^{-1}\left(e^{-\pi i y \cdot B^{-1}Ay} f \right) \right)_{B^{-1}} \right\|_{W(\mathcal{F}L^p,L^q)} \\
&\lesssim |\det B|^{1/q-1/p-1/2} (\det(B^*B+I))^{1/2} |\det(I+iDB^{-1})|^{1/2} \\
&\qquad \cdot \left\| \mathcal{F}^{-1}\left(e^{-\pi i y \cdot B^{-1}Ay} f \right) \right\|_{W(\mathcal{F}L^p,L^q)} \\
&\lesssim |\det B|^{1/q-1/p-1/2} (\det(B^*B+I))^{1/2} |\det(I+iDB^{-1})|^{1/2} \\
&\qquad \cdot \| e^{-\pi i y \cdot B^{-1}Ay} f \|_{W(\mathcal{F}L^q,L^p)} \\
&\lesssim \alpha(\mathcal{A},p,q) \|f\|_{W(\mathcal{F}L^q,L^p)}
\end{aligned}
$$

with $\alpha(\mathcal{A},p,q)$ given by (26).

(ii) In this case, we use formula (16). Then, proceeding likewise the case *(i)*, we majorize as follows:

$$
\begin{aligned}
\|\mu(\mathcal{A})f\|_{W(\mathcal{F}L^1,L^\infty)} &= |\det B|^{-1/2} \left\| e^{-\pi i x \cdot CA^{-1}x} \left(e^{-\pi i y \cdot B^{-1}Ay} * f \right)(A^{-1}x) \right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\leq |\det B|^{-1/2} \| e^{-\pi i x \cdot CA^{-1}x} \|_{W(\mathcal{F}L^1,L^\infty)} \\
&\qquad \cdot \left\| \left(e^{-\pi i y \cdot B^{-1}Ay} * f \right)_{A^{-1}} \right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim |\det B|^{-1/2} |\det A|^{-1} (\det(A^*A+I))^{1/2} |\det(I+iCA^{-1})|^{1/2} \\
&\qquad \cdot \left\| e^{-\pi i y \cdot B^{-1}Ay} * f \right\|_{W(\mathcal{F}L^1,L^\infty)} \\
&\lesssim \beta(\mathcal{A}) \|f\|_{W(\mathcal{F}L^\infty,L^1)},
\end{aligned}
$$

where the last raw is due to (7), with $\beta(\mathcal{A})$ defined in (28). $\qquad\square$

Proof of Theorem 4.2. The proof uses the same arguments as in Theorem 4.1. Here, the estimate (20) is replaced by (24). Besides, the index relation (22) is applied in

the final step. In details,

$$\|\mu(\mathcal{A})f\|_{W(\mathcal{F}L^p, L^q)} \leq |\det B|^{-1/2} \|e^{-\pi i x \cdot DB^{-1}x}\|_{W(\mathcal{F}L^1, L^\infty)}$$

$$\cdot \left\|\left(\mathcal{F}^{-1}\left(e^{-\pi i y \cdot B^{-1}Ay}f\right)\right)_{B^{-1}}\right\|_{W(\mathcal{F}L^p, L^q)}$$

$$\lesssim \prod_{j=1}^{d} |\lambda_j|^{-1/2} |\det(I + iDB^{-1})(I + iB^{-1}A)|^{1/2}$$

$$\cdot \prod_{j=1}^{d}(\max\{1, |\lambda_j|^{-1}\})^{\mu_1(p',q')}(\min\{1, |\lambda_j|^{-1}\})^{\mu_2(p',q')}\|f\|_{W(\mathcal{F}L^q, L^p)}$$

$$= |\det(B + iD)(B + iA)|^{1/2}$$

$$\cdot \prod_{j=1}^{d}(\max\{1, |\lambda_j|\})^{\mu_1(p,q)-1/2}(\min\{1, |\lambda_j|\})^{\mu_2(p,q)-1/2}\|f\|_{W(\mathcal{F}L^q, L^p)},$$

that is case *(i)*. Case *(ii)* indeed is not an improvement of (27) but is just (27) rephrased in terms of the eigenvalues of A. $\square$

Remark 4.3. The above theorems require the condition $\det B \neq 0$. However, in some special cases with $\det B = 0$, the previous results can still be used to obtain estimates between Wiener amalgam spaces. For example, if $\mathcal{A} = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$, with $C = C^*$, then $\mu(\mathcal{A})f(x) = \pm e^{-\pi i C x \cdot x}f(x)$ (see (13)), so that, for every $1 \leq p, q \leq \infty$, Proposition 2.5 and the estimate (33) give

$$\|\mu(\mathcal{A})f\|_{W(\mathcal{F}L^p, L^q)} \lesssim \prod_{j=1}^{d}(1 + \lambda_j^2)^{1/4}\|f\|_{W(\mathcal{F}L^p, L^q)},$$

where the λ_j's are the eigenvalues of C (incidentally, this estimate was already shown in [1, 2, 7]).

5. APPLICATIONS TO THE SCHRÖDINGER EQUATION

In this section we apply the previous results to the analysis of the Cauchy problem of Schrödinger equations with quadratic Hamiltonians, i.e.

$$(35) \qquad \begin{cases} i\dfrac{\partial u}{\partial t} + H_{\mathcal{A}}u = 0 \\ u(0, x) = u_0(x), \end{cases}$$

where $H_{\mathcal{A}}$ is the Weyl quantization of a quadratic form on the phase space $\mathbb{R}^{2d}$, defined from a matrix $\mathcal{A}$ in the Lie algebra $\mathfrak{sp}(d, \mathbb{R})$ of the symplectic group as follows (see [16] and [9]).

Any given matrix $\mathcal{A} \in \mathfrak{sp}(d, \mathbb{R})$ defines a quadratic form $\mathcal{P}_\mathcal{A}(x, \xi)$ in $\mathbb{R}^{2d}$ via the formula

$$\mathcal{P}_\mathcal{A}(x, \xi) = -\frac{1}{2}{}^t(x, \xi)\mathcal{A}\mathcal{J}(x, \xi),$$

where, as usual, $\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ (notice that $\mathcal{A}\mathcal{J}$ is symmetric). Explicitly, if $\mathcal{A} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{sp}(d, \mathbb{R})$ then

(36) $$P_\mathcal{A}(x, \xi) = \frac{1}{2}\xi \cdot B\xi - \xi \cdot Ax - \frac{1}{2}x \cdot Cx.$$

From the Weyl quantization, the quadratic polynomial $P_\mathcal{A}$ in (36) corresponds to the Weyl operator $\mathcal{P}_\mathcal{A}^w(D, X)$ defined by

$$2\pi\mathcal{P}_\mathcal{A}^w(D, X) = -\frac{1}{4\pi}\sum_{j,k=1}^d B_{j,k}\frac{\partial^2}{\partial x_j \partial x_k} + i\sum_{j,k=1}^d A_{j,k}x_k\frac{\partial}{\partial x_j} + \frac{i}{2}\mathrm{Tr}(A) - \pi\sum_{j,k=1}^d C_{j,k}x_j, x_k.$$

The operator $H_\mathcal{A} := 2\pi\mathcal{P}_\mathcal{A}^w(D, X)$ is called the Hamiltonian operator.

The evolution operator for (35) is related to the metaplectic representation via the following key formula

$$e^{itH_\mathcal{A}} = \mu(e^{t\mathcal{A}}).$$

Consequently, Theorems 4.1 and 4.2 can be used in the study of fixed-time estimates for the solution $u(t) = e^{itH_\mathcal{A}}u_0$ to (35).

As an example, consider the matrix $\mathcal{A} = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \in \mathfrak{sp}(d, \mathbb{R})$, with $B = B^*$.

Then the Hamiltonian operator is $H_\mathcal{A} = -\frac{1}{4\pi}B\nabla \cdot \nabla$ and $e^{it\mathcal{A}} = \begin{pmatrix} I & tB \\ 0 & I \end{pmatrix} \in Sp(d, \mathbb{R})$.

Fix $t \neq 0$. If $\det B \neq 0$, and B has eigenvalues $\lambda_1, \ldots, \lambda_d$, then the expression of $\beta'(e^{it\mathcal{A}})$ in (32) is given by

$$\beta'(e^{it\mathcal{A}}) = 2^{d/4}|\det tB|^{-1}|\det tB + iI)|^{1/2} = 2^{d/4}\prod_{j=1}^d \left(\frac{1 + t^2\lambda_j^2}{t^4\lambda_j^4}\right)^{1/4}.$$

Consequently, the fixed-time estimate (31) is

$$\|e^{itH_\mathcal{A}}f\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \prod_{j=1}^d \left(\frac{1 + t^2\lambda_j^2}{t^4\lambda_j^4}\right)^{1/4} \|f\|_{W(\mathcal{F}L^\infty, L^1)},$$

which generalizes the dispersive estimate in [7], corresponding to $B = I$.

In the next two sections we present new fixed-time estimates, and also Strichartz estimates, in the cases of the Hamiltonian $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta + \pi|x|^2$ and $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta - \pi|x|^2$.

5.1. Schrödinger equation with Hamiltonian $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta + \pi|x|^2$.

Here we consider the Cauchy problem (35) with the Hamiltonian $H_{\mathcal{A}}$ corresponding to the matrix $\mathcal{A} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \in \mathfrak{sp}(d,\mathbb{R})$, namely $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta + \pi|x|^2$. As a consequence of the estimates proved in the previous section we obtain the following fixed-time estimates.

Proposition 5.1. *For $2 \leq r \leq \infty$, we have the fixed-time estimates*

$$(37) \qquad \|e^{itH_{\mathcal{A}}}u_0\|_{W(\mathcal{F}L^{r'},L^r)} \lesssim |\sin t|^{-2d\left(\frac{1}{2}-\frac{1}{r}\right)}\|u_0\|_{W(\mathcal{F}L^r,L^{r'})}.$$

Proof. The symplectic matrix $e^{t\mathcal{A}}$ reveals to be $e^{t\mathcal{A}} = \begin{pmatrix} (\cos t)I & (\sin t)I \\ (-\sin t)I & (\cos t)I \end{pmatrix}$.

First, using the estimate (31) we get

$$(38) \qquad \|e^{itH_{\mathcal{A}}}u_0\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim |\sin t|^{-d}|\cos t|^{-\frac{3}{2}d}\|u_0\|_{W(\mathcal{F}L^\infty,L^1)}.$$

On the other hand, the estimate (29), for $p=1, q=\infty$, reads

$$(39) \qquad \|e^{itH_{\mathcal{A}}}u_0\|_{W(\mathcal{F}L^1,L^\infty)} \lesssim |\sin t|^{-5d/2}\|u_0\|_{W(\mathcal{F}L^\infty,L^1)}.$$

Since $\min\{|\sin t|^{-d}|\cos t|^{-\frac{3}{2}d}, |\sin t|^{-5d/2}\} \asymp |\sin t|^{-d}$, we obtain (37) for $r=\infty$, which is the dispersive estimate.

The estimates (37) for $2 \leq r \leq \infty$ follow by complex interpolation from the dispersive estimate and the $L^2 - L^2$ estimate

$$(40) \qquad \|e^{itH_{\mathcal{A}}}f\|_{L^2} = \|f\|_{L^2}.$$

$\square$

The Strichartz estimates for the solutions to (35) are detailed as follows.

Theorem 5.2. *Let $T > 0$ and $4 < q, \tilde{q} \leq \infty$, $2 \leq r, \tilde{r} \leq \infty$, such that*

$$(41) \qquad \frac{2}{q} + \frac{d}{r} = \frac{d}{2},$$

and similarly for $\tilde{q}, \tilde{r}$. Then we have the homogeneous Strichartz estimates

$$(42) \qquad \|e^{itH_{\mathcal{A}}}u_0\|_{L^{q/2}([0,T])W(\mathcal{F}L^{r'},L^r)_x} \lesssim \|u_0\|_{L_x^2},$$

the dual homogeneous Strichartz estimates

$$(43) \qquad \|\int_0^T e^{-isH_{\mathcal{A}}}F(s)\,ds\|_{L^2} \lesssim \|F\|_{L^{(\tilde{q}/2)'}([0,T])W(\mathcal{F}L^{\tilde{r}},L^{\tilde{r}'})_x},$$

and the retarded Strichartz estimates

$$(44) \quad \| \int_{0 \le s < t} e^{i(t-s)H_{\mathcal{A}}} F(s) \, ds \|_{L^{q/2}([0,T])W(\mathcal{F}L^{r'},L^r)_x} \lesssim \|F\|_{L^{(\tilde{q}/2)'}([0,T])W(\mathcal{F}L^{\tilde{r}},L^{\tilde{r}'})_x}.$$

Consider then the endpoint $P := (4, 2d/(d-1))$. *For* $(q,r) = P$, $d > 1$, *we have*

$$(45) \quad \|e^{itH_{\mathcal{A}}} u_0\|_{L^2([0,T])W(\mathcal{F}L^{r',2},L^r)_x} \lesssim \|u_0\|_{L_x^2},$$

$$(46) \quad \| \int_0^T e^{-isH_{\mathcal{A}}} F(s) \, ds \|_{L^2} \lesssim \|F\|_{L^2([0,T])W(\mathcal{F}L^{r,2},L^{r'})_x}.$$

The retarded estimates (44) *still hold with* (q,r) *satisfying* (41), $q > 4, r \ge 2$, $(\tilde{q}, \tilde{r}) = P$, *if one replaces* $\mathcal{F}L^{\tilde{r}'}$ *by* $\mathcal{F}L^{\tilde{r}',2}$. *Similarly it holds for* $(q,r) = P$ *and* $(\tilde{q}, \tilde{r}) \ne P$ *as above if one replaces* $\mathcal{F}L^{r'}$ *by* $\mathcal{F}L^{r',2}$. *It holds for both* $(p,r) = (\tilde{p}, \tilde{r}) = P$ *if one replaces* $\mathcal{F}L^{r'}$ *by* $\mathcal{F}L^{r',2}$ *and* $\mathcal{F}L^{\tilde{r}'}$ *by* $\mathcal{F}L^{\tilde{r}',2}$.

In the previous theorem the bounds may depend on T.

Proof. The arguments are essentially the ones in [7, 24]. For the convenience of the reader, we present the guidelines of the proof.

Due to the property group of the evolution operator $e^{itH_{\mathcal{A}}}$, we can limit ourselves to the case $T = 1$. Indeed, observe that, if (42) holds for a given $T > 0$, it holds for any $0 < T' \le T$ as well, so that it suffices to prove (42) for $T = N$ integer. Since

$$\|e^{itH_{\mathcal{A}}} u_0\|_{L^{q/2}([0,N])W(\mathcal{F}L^{r'},L^r)_x}^{\frac{q}{2}} = \sum_{k=0}^{N-1} \|e^{itH_{\mathcal{A}}} e^{ikH_{\mathcal{A}}} u_0\|_{L^{q/2}([0,1])W(\mathcal{F}L^{r'},L^r)_x}^{\frac{q}{2}},$$

the $T = N$ case is reduced to the $T = 1$ case by using (42) for $T = 1$ and the conservation law (40). The other estimates can be treated analogously. Whence from now on $T = 1$.

Consider first the non-endpoint case. Set $U(t) = \chi_{[0,1]}(t) e^{itH_{\mathcal{A}}}$. For $2 \le r \le \infty$, using relation (37), we get

$$(47) \quad \|U(t)(U(s))^* f\|_{W(\mathcal{F}L^{r'},L^r)} \lesssim |t-s|^{-2d\left(\frac{1}{2}-\frac{1}{r}\right)} \|f\|_{W(\mathcal{F}L^r,L^{r'})}.$$

By the TT^* method[1] (see, e.g., [17, Lemma 2.1] or [28, page 353]) the estimate (42) is equivalent to

$$(48) \quad \| \int U(t)(U(s))^* F(s) \, ds \|_{L_t^{q/2} W(\mathcal{F}L^{r'},L^r)_x} \lesssim \|F\|_{L_t^{(q/2)'} W(\mathcal{F}L^{\tilde{r}},L^{r'})_x}.$$

---

[1]This duality argument is generally established for L^p spaces. Its use for Wiener amalgam spaces is similarly justified thanks to the duality defined by the Hölder-type inequality [7]:

$$|\langle F, G \rangle_{L_t^2 L_x^2}| \le \|F\|_{W(L^s,L^q)_t W(\mathcal{F}L^{r'},L^r))_x} \|G\|_{W(L^{s'},L^{q'})_t W(\mathcal{F}L^r,L^{r'})_x}.$$

The estimate above is attained by applying Minkowski's inequality and the Hardy-Littlewood-Sobolev inequality (5) to the estimate (47). The dual homogeneous estimates (43) follow by duality. Finally, the retarded estimates (44), with $(1/q, 1/r)$, $(1/\tilde{q}, 1/\tilde{r})$ and $(1/\infty, 1/2)$ collinear, follow by complex interpolation from the three cases $(\tilde{q}, \tilde{r}) = (q, r)$, $(q, r) = (\infty, 2)$ and $(\tilde{q}, \tilde{r}) = (\infty, 2)$, which in turns are a consequence, of (48) (with $\chi_{s<t}F$ in place of F), (43) (with $\chi_{s<t}F$ in place of F) and the duality argument, respectively.

We are left to the endpoint case: $(q, r) = (2, 2d/(d-1))$. The estimate (45) is equivalent to the bilinear estimate

$$\left| \iint \langle (U(s))^*F(s), (U(t))^*G(t) \rangle \, ds \, dt \right| \lesssim \|F\|_{L_t^2 W(\mathcal{F}L^{r,2}, L^{r'})_x} \|G\|_{L_t^2 W(\mathcal{F}L^{r,2}, L^{r'})_x}.$$

By symmetry, it is enough to prove

$$(49) \qquad |T(F, G)| \lesssim \|F\|_{L_t^2 W(\mathcal{F}L^{r,2}, L^{r'})_x} \|G\|_{L_t^2 W(\mathcal{F}L^{r,2}, L^{r'})_x},$$

where

$$T(F, G) = \iint_{s<t} \langle (U(s))^*F(s), (U(t))^*G(t) \rangle \, ds \, dt.$$

To this aim, $T(F, G)$ is decomposed dyadically as $T = \sum_{j \in \mathbb{Z}} T_j$, with

$$(50) \qquad T_j(F, G) = \iint_{t-2^{j+1}<s\leq t-2^j} (U(s))^*F(s), (U(t))^*G(t) \rangle \, ds \, dt.$$

By resorting on (43) one can prove exactly as in [24, Lemma 4.1] the following estimates:

$$(51) \qquad |T_j(F, G)| \lesssim 2^{-j\beta(a,b)} \|F\|_{L_t^2 W(\mathcal{F}L^a, L^{a'})} \|G\|_{L_t^2 W(\mathcal{F}L^b, L^{b'})},$$

for $(1/a, 1/b)$ in a neighborhood of $(1/r, 1/r)$, with $\beta(a, b) = d - 1 - \frac{d}{a} - \frac{d}{b}$.

The estimate (49) is achieved by means of a real interpolation result, detailed in [24, Lemma 6.1], and applied to the vector-valued bilinear operator $T = (T_j)_{j \in \mathbb{Z}}$. Here, however, we must observe that, if $A_k = L_t^2 W(\mathcal{F}L^{a_k}, L^{a_k'})_x$, $k = 0, 1$, and θ_0 fulfills $1/r = (1 - \theta_0)/a_0 + \theta_0/a_1$, then

$$L_t^2 W(\mathcal{F}L^{r,2}, L^{r'})_x \subset (A_0, A_1)_{\theta_0, 2}.$$

The above inclusion follows by [37, Theorem 1.18.4, page 129] (with $p = p_0 = p_1 = 2$) and Proposition 2.3. This gives (45) and (46).

Consider now the endpoint retarded estimates. The case $(\tilde{q}, \tilde{r}) = (q, r) = P$ is exactly (49). The case $(\tilde{q}, \tilde{r}) = P$, $(q, r) \neq P$, can be obtained by a repeated use of Hölder's inequality to interpolate from the case $(\tilde{q}, \tilde{r}) = (q, r) = P$ and the case $(\tilde{q}, \tilde{r}) = P$, $(q, r) = (\infty, 2)$ (that is clear from (46)). Finally, the retarded estimate in the case $(q, r) = P$, $(\tilde{q}, \tilde{r}) \neq P$, follows by applying the arguments above to the adjoint operator $G \mapsto \int_{t>s} (U(t))^*U(s)G(t) \, dt$, which gives the dual estimate. □

5.2. Schrödinger equation with Hamiltonian $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta - \pi|x|^2$.

The Hamiltonian operator $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta - \pi|x|^2$ corresponds to the matrix $\mathcal{A} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \in \mathfrak{sp}(d, \mathbb{R})$. In this case, $e^{t\mathcal{A}} = \begin{pmatrix} (\cosh t)I & (\sinh t)I \\ (\sinh t)I & (\cosh t)I \end{pmatrix} \in Sp(d, \mathbb{R})$.

Fixed-time estimates for $H_{\mathcal{A}}$ are as follows.

Proposition 5.3. *For $2 \le r \le \infty$,*

$$(52) \qquad \|e^{itH_{\mathcal{A}}}u_0\|_{W(\mathcal{F}L^{r'}, L^r)} \lesssim \left(\frac{1 + |\sinh t|}{\sinh^2 t}\right)^{d\left(\frac{1}{2} - \frac{1}{r}\right)} \|u_0\|_{W(\mathcal{F}L^r, L^{r'})}.$$

Proof. The estimate (31) yields the dispersive estimate

$$(53) \qquad \|e^{itH_{\mathcal{A}}}u_0\|_{W(\mathcal{F}L^1, L^\infty)} \lesssim \left(\frac{1 + |\sinh t|}{\sinh^2 t}\right)^{\frac{d}{2}} \|u_0\|_{W(\mathcal{F}L^\infty, L^1)}.$$

(Observe that (29), with $p = 1$, $q = \infty$, gives a bound worse than (53)).

The estimates (52) follow by complex interpolation between the dispersive estimate (53) and the conservation law (40). □

We can now establish the corresponding Strichartz estimates.

Theorem 5.4. *Let $4 < q, \tilde{q} \le \infty$, $2 \le r, \tilde{r} \le \infty$, such that*

$$(54) \qquad \frac{2}{q} + \frac{d}{r} = \frac{d}{2},$$

and similarly for $\tilde{q}, \tilde{r}$. Then we have the homogeneous Strichartz estimates

$$(55) \qquad \|e^{itH_{\mathcal{A}}}u_0\|_{W(L^{q/2}, L^2)_t W(\mathcal{F}L^{r'}, L^r)_x} \lesssim \|u_0\|_{L_x^2},$$

the dual homogeneous Strichartz estimates

$$(56) \qquad \left\| \int e^{-isH_{\mathcal{A}}} F(s)\, ds \right\|_{L^2} \lesssim \|F\|_{W(L^{(\tilde{q}/2)'}, L^2)_t W(\mathcal{F}L^{\tilde{r}}, L^{\tilde{r}'})_x},$$

and the retarded Strichartz estimates

$$(57) \quad \left\| \int_{s<t} e^{i(t-s)H_{\mathcal{A}}} F(s)\, ds \right\|_{W(L^{q/2}, L^2)_t W(\mathcal{F}L^{r'}, L^r)_x} \lesssim \|F\|_{W(L^{(\tilde{q}/2)'}, L^2)_t W(\mathcal{F}L^{\tilde{r}}, L^{\tilde{r}'})_x}.$$

Consider then the endpoint $P := (4, 2d/(d-1))$. For $(q, r) = P$, $d > 1$, we have

$$(58) \qquad \|e^{itH_{\mathcal{A}}}u_0\|_{L_t^2 W(\mathcal{F}L^{r', 2}, L^r)_x} \lesssim \|u_0\|_{L_x^2},$$

$$(59) \qquad \left\| \int e^{-isH_{\mathcal{A}}} F(s)\, ds \right\|_{L^2} \lesssim \|F\|_{L_t^2 W(\mathcal{F}L^{r, 2}, L^{r'})_x}.$$

The retarded estimates (57) still hold with (q, r) satisfying (54), $q > 4, r \ge 2$, $(\tilde{q}, \tilde{r}) = P$, if one replaces $\mathcal{F}L^{\tilde{r}'}$ by $\mathcal{F}L^{\tilde{r}', 2}$. Similarly it holds for $(q, r) = P$ and

$(\tilde{q},\tilde{r}) \neq P$ as above if one replaces $\mathcal{F}L^{r'}$ by $\mathcal{F}L^{r',2}$. It holds for both $(p,r) = (\tilde{p},\tilde{r}) = P$ if one replaces $\mathcal{F}L^{r'}$ by $\mathcal{F}L^{r',2}$ and $\mathcal{F}L^{\tilde{r}'}$ by $\mathcal{F}L^{\tilde{r}',2}$.

Proof. Let us first prove (55). By the TT^* method it suffices to prove

$$(60) \quad \left\| \int e^{i(t-s)H_{\mathcal{A}}} F(s)\, ds \right\|_{W(L^{q/2},L^2)_t W(\mathcal{F}L^{r'},L^r)_x} \lesssim \|F\|_{W(L^{(q/2)'},L^2)_t W(\mathcal{F}L^r,L^{r'})_x}.$$

For $0 < \alpha < 1/2$, let $\phi_\alpha(t) = |\sinh t|^{-\alpha} + |\sinh t|^{-2\alpha}$, $t \in \mathbb{R}$, $t \neq 0$. A direct computation shows that $\phi_\alpha \in W(L^{1/(2\alpha),\infty}, L^1)$. Since $L^1 * L^2 \hookrightarrow L^2$ (Young's Inequality) and $L^{\left(\frac{1}{\alpha}\right)'} * L^{\frac{1}{2\alpha},\infty} \hookrightarrow L^{\frac{1}{\alpha}}$ (Proposition 2.1), Lemma 2.1 (i) gives the convolution relation

$$(61) \qquad\qquad \|F * \phi_\alpha\|_{W(L^{1/\alpha},L^{2/\alpha})} \lesssim \|F\|_{W(L^{(1/\alpha)'},L^{(2/\alpha)'})}.$$

Fix now $\alpha = d(1/2 - 1/r) = 2/q$; then, by (52), (61) and Minkowski's Inequality,

$$\left\| \int e^{i(t-s)H_{\mathcal{A}}} F(s)\, ds \right\|_{W(L^{q/2},L^2)_t W(\mathcal{F}L^{r'},L^r)_x}$$

$$\leq \left\| \int \|e^{i(t-s)H_{\mathcal{A}}} F(s)\|_{W(\mathcal{F}L^{r'},L^r)_x}\, ds \right\|_{W(L^{q/2},L^2)_t}$$

$$\lesssim \left\| \|F(t)\|_{W(\mathcal{F}L^{r'},L^r)_x} * \phi_\alpha(t) \right\|_{W(L^{q/2},L^2)_t}$$

$$\lesssim \|F\|_{W(L^{(q/2)'},L^2)_t W(\mathcal{F}L^r,L^{r'})_x}.$$

This proves (60) and whence (55). The estimate (56) follows from (55) by duality. The proof of (57) is analogous to (44) in Theorem 5.2.

For the endpoint case one can repeat essentially verbatim the arguments in the proof of Theorem 5.2, upon setting $U(t) = e^{itH_{\mathcal{A}}}$. To avoid repetitions, we omit the details (see also the proof of [7, Theorem 1.2]). $\qquad\square$

5.3. Comparison with the classical estimates in Lebesgue spaces.

Here we compare the above estimates with the classical ones between Lebesgue spaces. For the convenience of the reader we recall the following very general result by Keel and Tao [24, Theorem 1.2].

Given $\sigma > 0$, we say that an exponent pair (q,r) is *sharp σ-admissible* if $1/q + \sigma/r = \sigma/2$, $q \geq 2, r \geq 2$, $(q,r,\sigma) \neq (2,\infty,1)$.

Theorem 5.5. *Let $(X,\mathcal{S},\mu)$ be a σ-finite measured space, and $U : \mathbb{R} \to B(L^2(X,\mathcal{S},\mu))$ be a weakly measurable map satisfying, for some $\sigma > 0$,*

$$\|U(t)f\|_{L^2} \lesssim \|u\|_{L^2}, \quad t \in \mathbb{R},$$

and

$$\|U(s)U(t)^* f\|_{L^\infty} \lesssim |t-s|^{-\sigma} \|f\|_{L^1}, \quad t,s \in \mathbb{R}.$$

Then for every sharp σ-admissible pairs (q, r), $(\tilde{q}, \tilde{r})$, one has

$$\|U(t)f\|_{L_t^q L_x^r} \lesssim \|f\|_{L^2},$$

$$\|\int U(s)^* F(s)\,ds\|_{L^2} \lesssim \|F\|_{L_t^{q'} L_x^{r'}},$$

$$\|\int_{s<t} U(t)U(s)^* F(s)\,ds\|_{L_t^q L_x^r} \lesssim \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}.$$

First we fix the attention to the case of the Hamiltonian $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta + \pi|x|^2$. One has the following explicit formula for $e^{itH_{\mathcal{A}}}u_0 = \mu(e^{it\mathcal{A}})u_0$ in (14):

$$e^{itH_{\mathcal{A}}}u_0 = i^{d/2}(\sin t)^{-d/2}\int e^{-\pi i(\cot t)(|x|^2+|y|^2)+2\pi i(\operatorname{cosec} t)y\cdot x}u_0(y)\,dy.$$

Hereby it follows immediately the dispersive estimate

(62) $$\|e^{itH_{\mathcal{A}}}u_0\|_{L^\infty} \leq |\sin t|^{-d/2}\|u\|_{L^1}.$$

Notice that (3) (i.e. (37) with $r = \infty$) represents an improvement of (62) for every fixed $t \neq 0$, since $L^1 \hookrightarrow W(\mathcal{F}L^\infty, L^1)$ and $W(\mathcal{F}L^1, L^\infty) \hookrightarrow L^\infty$. However, as might be expected, the bound on the norm in (3) becomes worse than that in (62) as $t \to k\pi$, $k \in \mathbb{Z}$.

As a consequence of (62), Theorem 5.5 with $U(t) = e^{itH_{\mathcal{A}}}\chi_{[0,1]}(t)$ and $\sigma = d/2$, and the group property of the operator $e^{itH_{\mathcal{A}}}$ (as in the proof of Theorem 5.2 above) one deduce, for example, the homogeneous Strichartz estimate

(63) $$\|e^{itH_{\mathcal{A}}}u_0\|_{L^q([0,T])L_x^r} \lesssim \|u_0\|_{L_x^2},$$

for every pair (q, r) satisfying $2/q + d/r = d/2$, $q \geq 2, r \geq 2$, $(q, r, d) \neq (2, \infty, 2)$. These estimates were also obtained recently in [25] by different methods.

Hence, one sees that (42) predicts, for the solution to (35), a better local spatial regularity than (63), but just after averaging on $[0, T]$ by the $L^{q/2}$ norm, which is smaller than the L^q norm.

We now consider the case of the Hamiltonian $H_{\mathcal{A}} = -\frac{1}{4\pi}\Delta - \pi|x|^2$. The dispersive estimate here reads

(64) $$\|e^{itH_{\mathcal{A}}}u_0\|_{L^\infty(\mathbb{R}^d)} \leq |\sinh t|^{-d/2}\|u_0\|_{L^1(\mathbb{R}^d)}.$$

This estimate follows immediately from the explicit expression of $e^{itH_{\mathcal{A}}}u_0 = \mu(e^{it\mathcal{A}})u_0$ in (14):

$$e^{itH_{\mathcal{A}}}u_0 = i^{d/2}(\sinh t)^{-d/2}\int e^{-\pi i(\coth t)(|x|^2+|y|^2)+2\pi i(\operatorname{cosech} t)y\cdot x}u_0(y)\,dy.$$

The corresponding Strichartz estimates between the Lebesgue spaces read

(65) $$\|e^{itH_{\mathcal{A}}}u_0\|_{L_t^q L_x^r} \lesssim \|u_0\|_{L_x^2},$$

for $q \geq 2$, $r \geq 2$, with $2/q + d/r = d/2$, $(q, r, d) \neq (2, \infty, 2)$. These estimates are the issues of Theorem 5.5 with $U(t) = e^{itH_A}$, and the dispersive estimate (64) (indeed, $|\sinh t|^{-d/2} \leq |t|^{-d/2}$). These estimates are to be compared with (52) (with $r = \infty$) and (55) respectively.

One can do the same remarks as in the previous case. In addition here one should observe that (55) displays a better time decay at infinity than the classical one (L^2 instead of L^r), for a norm, $\|u(t, \cdot)\|_{W(\mathcal{F}L^{r'}, L^r)}$, which is even bigger than L^r. Notice however that our range of exponents is restricted to $q \geq 4$.

ACKNOWLEDGEMENTS

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